SELECTION OF AN INITIAL APPROXIMATION IN AN ASYMPTOTIC REPRESENTATION OF THE SOLUTION OF THE PROBLEM OF A POINT-SOURCE THERMAL EXPLOSION IN A NONLINEARLY HEAT-CONDUCTING GAS

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Qualitative analysis shows [1] that the initial stage of a point-source thermal explosion in a homogeneous stationary gas is characterized by the predominance of radiative heat transfer. This finding is related to the nonlinear dependence of the coefficient of radiative heat transfer κ on the temperature of the gas T. The function $\kappa(T)$ can be assigned in power form $\kappa = \lambda/n\ T^{n-1}$, where λ is a dimensional constant and $n > 1$ is the nonlinearity exponent. In complete agreement with the qualitative conclusions in [1], the author of [2] found by the asymptotic method that for short periods of time after the explosion, radiative heat transfer occurs independently of the motion of the gas and completely determines it. Here, the occurrence of the shock wave in [2] is connected with the convergence of the asymptotic solution on the well-known self-similar solution for a thermal explosion in a non-heat-conducting gas [3]. Conversely, the experimental findings and qualitative analysis of the problem in [1, 4] indicate that an isothermal shock wave can occur within a finite period of time after a thermal explosion. The shock then separated from the region heated by radiation [1], while radiative heat transfer turns out to have a diminishing effect on its motion. The role of heat transfer is negligible far from the site of the explosion, and the motion of the shock becomes self-similar [3]. Using the example of a plane thermal explosion in a nonlinearly heat-conducting ideal gas for the case $n \gg 1$, here we propose an asymptotic representation of the solution of the above problem which will make it possible to analyze the generation of an isothermal shock wave.

1. Formulation of the Problem and Its Asymptotic Analysis. Let a quantity of thermal energy $Q = 2Q_0$ be instantaneously released at the moment of time $t = 0$ in the plane $x = 0$ in an infinite space filled with a stationary ideal gas having a density ρ_0, specific heat c_V, and temperature $T = 0$. It is convenient to take the following as characteristic parameters of the gas; ρ_0 is the initial density of the gas; $T_0 = [Q_0 R/\lambda a]^{2/(2n-1)}$ is the temperature (R is the gas constant, $a = [(n-1)/2n(n+1)]^{1/(n-1)}$), $W = \sqrt{RT_0}$ is the velocity, $L = \lambda T_0^{(n-3/2)}/c_V\rho_0\sqrt{R}$ is the length, $t_0 = L/W$ is the time. Then the problem of a point-source explosion is described in dimensionless variables by the following system of equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} = 0,\quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial T}{\partial x} + \frac{T}{\rho}\frac{\partial\rho}{\partial x} = 0, \tag{1.1}$$
$$\rho\frac{\partial T}{\partial t} + \rho u\frac{\partial T}{\partial x} + \frac{R}{c_V}T\rho\frac{\partial u}{\partial x} = \frac{\partial^2 T^n}{\partial x^2}$$

with the boundary and initial conditions

$$T = \partial T^n/\partial x = u = 0,\ \rho = 1 \quad \text{at} \quad x = \pm\infty,\ t > 0; \tag{1.2}$$

$$T = \delta(x),\ \rho = 1,\ u = 0 \quad \text{at} \quad t = 0,\ |x| < \infty. \tag{1.3}$$

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Here, $\rho(x, t)$, $u(x, t)$, $T(x, t)$ are the dimensionless density, velocity, and temperature of the gas; $\delta(x)$ is the Dirac delta function.

It was shown in [1] that one feature of the process at $n > 1$ is its localization in space, i.e., there exists a surface $[x] = x_f(t) < \infty$ such that $T = u = 0$, $\rho = 1$ at $|x| > x_f(x)$. Thus, in regard to continuity the boundary conditions (1.2) are transferred to the surface $|x| = x_f(t)$:

$$T = \partial T^n/\partial x = u = 0,\ \rho = 1 \quad \text{at} \quad |x| = x_f(t). \tag{1.4}$$

It should be noted that the function $x_f(t)$ is unknown and must be found during solution of the stated boundary-value problem.

If we ignore motion of the gas and assume that $u \equiv 0$, $\rho \equiv 1$, problem (1.1)-(1.4) reduces to the familiar problem of the proagation of heat from an instantaneous planar source [5]. The solution of the latter problem has the form

$$T(x, t) = t^{-\frac{1}{n-1}} x_f^{\frac{2}{n-1}} \chi(\eta), \quad x_f = ct^{\frac{1}{n+1}}, \tag{1.5}$$

where $\chi = a(1-\eta^2)^{\frac{1}{n-1}}$; $\eta = \frac{x}{x_f}$; $c = \left[\frac{1}{2} B\left(\frac{n}{n-1}, \frac{1}{2}\right)\right]^{-\frac{n-1}{n+1}}$. Then we can use the known temperature T to evaluate the rate of propagation of hydrodynamic perturbations, which should be close to the isothermal speed of sound $S = \sqrt{T} \sim t^{-1/2(n+1)}$. Comparing it with the characteristic rate of radiant heat transfer $dx_f/dt \sim t^{-n/(n+1)}$, in accordance with [1] it must be concluded that at $n > 1$ radiative heat transfer completely determines the process as $t \to 0$. It also follows from the comparison that in the case of plane symmetry, there is no similitude between the hydrodynamic processes and the process of radiative heat transfer at $n > 1$ and $t \to 0$ either. Thus there are no self-similar solutions here which are analogous to those constructed in [6].

We will assume that the solution of boundary-value problem (1.1)-(1.4) at the limit at $t \to 0$ continuously transforms into solution (1.5), and the initial conditions with respect to the dynamic variables $u = 0$, $\rho = 1$. To construct an asymptotic representation of the solution of problem (1.1-1.4) at $t \to 0$, it is convenient to change over to new independent variables $x, t \to \eta = x/x_f$ and new dependent variables

$$T = t^{-\frac{1}{n+1}} x_f^{\frac{2}{n-1}} \chi(\eta, t), \quad \rho = 1 + t^{\beta_1} x_f^{\beta_2} r(\eta, t),$$
$$u = t^{\alpha_1} x_f^{\alpha_2} v(\eta, t), \quad x_f = t^{\frac{1}{n+1}} c(t), \tag{1.6}$$

assuming that the functions $\chi(\eta, t)$, $r(\eta, t)$, $v(\eta, t)$, $c(t)$ and their derivatives have the order $O(1)$ at $t \to 0$. In the formulation of Eqs. (1.6), system of equations (1.1) contains the constants $\alpha_1 = (n-2)/(n-1)$, $\alpha_2 = (3-n)/(n-1)$, $\beta_1 = (2n-3)/(n-1)$, $\beta_2 = (2(2-n))/(n-1)$, while system (1.1) appears as follows in the new variables

$$\frac{2n-1}{n+1} r + t\frac{\partial r}{\partial t} - \frac{1}{n+1}\eta\frac{\partial r}{\partial \eta} + \frac{\partial v}{\partial \eta} + c^{\frac{2(2-n)}{n-1}} t^{\frac{2n-1}{n+1}} \frac{\partial rv}{\partial \eta} + \frac{2(2-n)}{n-1}\frac{t}{c} r\frac{dc}{dt} - \eta\frac{t}{c}\frac{dc}{dt}\frac{\partial r}{\partial \eta} = 0, \tag{1.7}$$

$$\frac{n-1}{n+1} v + t\frac{\partial v}{\partial t} - \frac{1}{n+1}\eta\frac{\partial v}{\partial \eta} + c^{\frac{2(2-n)}{n-1}} t^{\frac{2n-1}{n+1}} v\frac{\partial v}{\partial \eta} + \frac{\partial \chi}{\partial \eta} + c^{\frac{2(2-n)}{n-1}} t^{\frac{2n-1}{n+1}} \frac{\chi}{\rho}\frac{\partial r}{\partial \eta} + \frac{3-n}{n-1}\frac{t}{c} v\frac{dc}{dt} - \eta\frac{t}{c}\frac{dc}{dt}\frac{\partial v}{\partial \eta} = 0,$$

$$-\frac{1}{n+1}\chi - \frac{1}{n+1}\eta\frac{\partial \chi}{\partial \eta} + c^{\frac{2(2-n)}{n-1}} t^{\frac{2n-1}{n+1}} v\frac{\partial \chi}{\partial \eta} + (k-1)c^{\frac{2(2-n)}{n-1}} t^{\frac{2n-1}{n+1}} \chi\frac{\partial v}{\partial \eta} + t\frac{\partial \chi}{\partial t} + \frac{2}{n-1}\frac{t}{c}\chi\frac{dc}{dt} - \eta\frac{t}{c}\frac{dc}{dt}\frac{\partial \chi}{\partial \eta} = \frac{1}{\rho}\frac{\partial^2 \chi^n}{\partial \eta^2}.$$

System (1.7) must be augmented by boundary conditions which follow from Eqs. (1.3) and (1.4). The above-formulated problem is symmetrical relative to $\eta = 0$. Thus, we will henceforth limit ourselves to the region $\eta > 0$. Then instead of conditions (1.4) we can write boundary conditions of the following form in the new variables

$$\chi = \partial\chi^n \partial\eta = v = 0,\ r = 0 \quad \text{at} \quad \eta = 1,\ t > 0; \tag{1.8}$$

$$\partial\chi/\partial\eta = v = 0 \quad \text{at} \quad \eta = 0,\ t > 0. \tag{1.9}$$

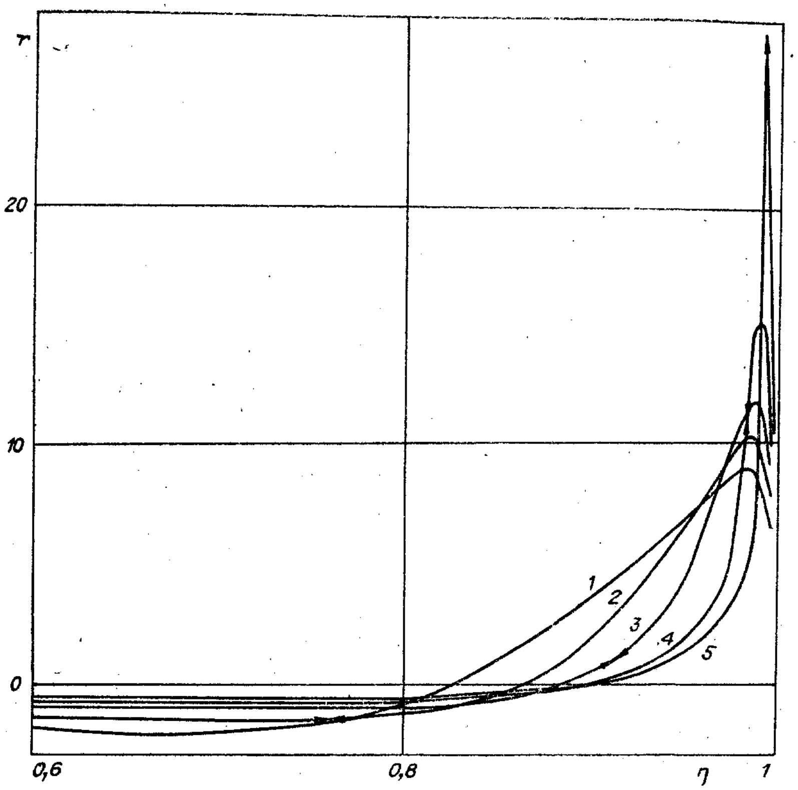

Fig. 1

Initial conditions (1.3) turn out to be satisfied due to the above assumptions made regarding the character of the solutions at $t \to 0$.

If we ignore terms of the order $O(t)$ in Eqs. (1.7), then the function $\chi(\eta, t) = \chi(\eta)$ and $c(t) = \text{const}$ are determined independently in the form (1.5), while for v and r we obtain the relations

$$\frac{2n-1}{n+1} r - \frac{1}{n+1} \eta \frac{\partial r}{\partial \eta} + \frac{\partial v}{\partial \eta} = 0, \quad \frac{n-1}{n+1} v - \frac{1}{n+1} \eta \frac{\partial v}{\partial \eta} + \frac{\partial \chi}{\partial \eta} = 0, \tag{1.10}$$

which coincide with the first approximation in [2]. In this approximation, the functions $r(\eta, t) = r(\eta)$, $v(\eta, t) = v(\eta)$ are determined from (1.8) and (1.10) in quadratures, while condition (1.9) is satisfied automatically (see [2]).

Equations (1.10) are singularly perturbed at $n \to \infty$. Thus, it is necessary to assume [7, 8] that if $n >> 1$, then when $\eta \to 1$ the functions $v(\eta)$ and $r(\eta)$ have the character of a boundary layer within which the derivatives $\partial r/\partial \eta \gg 1$, $\partial v/\partial \eta \gg 1$. Following [7, 8] to explain the features of the behavior of the solutions of problem (1.7-1.9) at $n >> 1$ and $\eta \to 1$, we change the scales of the independent variable $\eta = 1 - \varepsilon^{\omega}\eta^*$ and dependence variables $r = \varepsilon^{\beta} r^*$, $v = \varepsilon^{\alpha} v^*$, where $\varepsilon = 1/n << 1$. By inserting these relations into system (1.7) we find that $\omega = 1$, $\beta = -1$, $\alpha = 0$. The equations in the altered variables are not presented here due to their awkwardness. It turns out that $\eta^* = O(1)$ within the boundary layer, and it is necessary to the terms added to Eqs. (1.10) even at $t = O(1/n) << 1$. Thus, if as the first approximation we choose a relation which describes the solution of the stated problem with an accuracy that is uniform with respect to $t = O(1/n) << 1$ throughout the region $0 < \eta < 1$, then in the former variables we obtain

$$\frac{2n-1}{n+1} r - \frac{1}{n+1} \eta \frac{\partial r}{\partial \eta} + \frac{\partial v}{\partial \eta} = A + O(t^2), \tag{1.11}$$

$$\frac{n-1}{n+1} v - \frac{1}{n+1} \eta \frac{\partial v}{\partial \eta} + \frac{\partial \chi}{\partial \eta} + \frac{t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}}}{1+D} \chi \frac{\partial r}{\partial \eta} = B + O(t^2).$$

Here, $A = -t\frac{\partial r}{\partial t} - t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}} \frac{\partial rv}{\partial \eta}$, $B = -t\frac{\partial v}{\partial t} - t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}} v \frac{\partial v}{\partial \eta}$, $D = t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}} r$ has the asymptotic

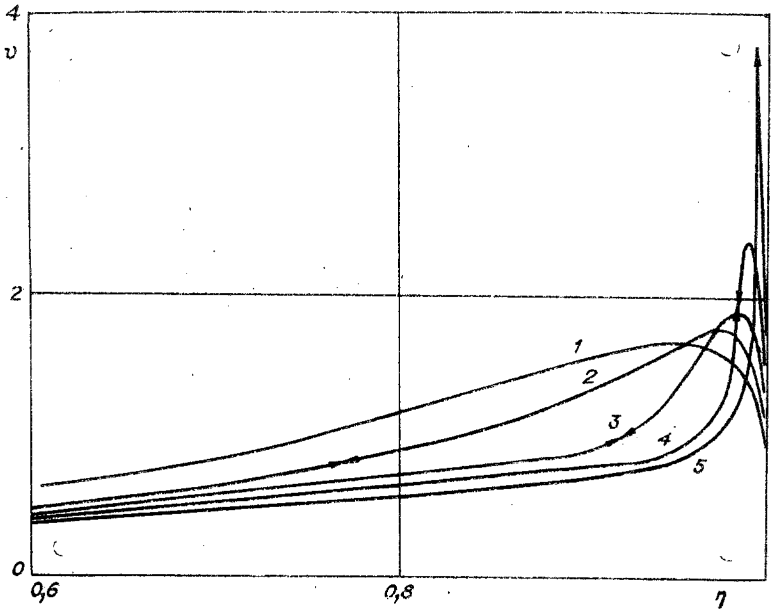

Fig. 2

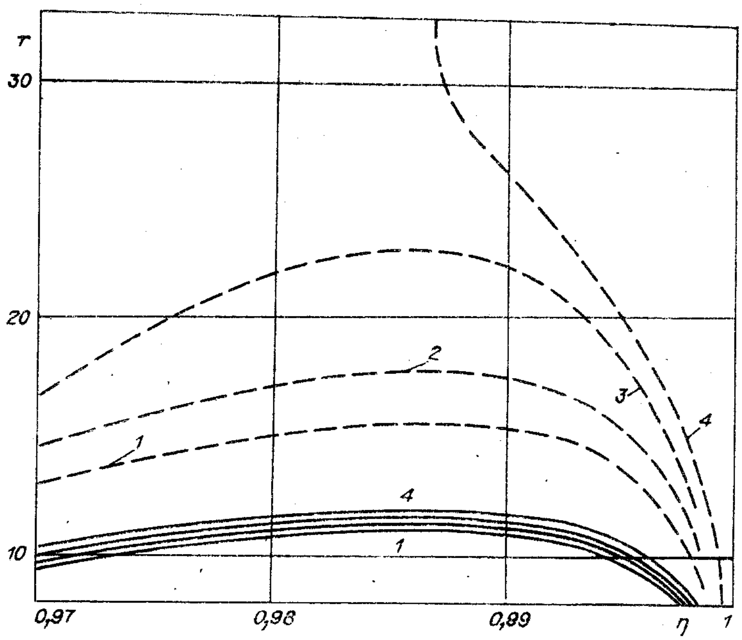

Fig. 3

order O(t). Equations (1.5) remain valid in this approximation for χ and c. Thus, the problem reduces to the study of Eqs. (1.11) with boundary conditions (1.8) and (1.9).

2. Numerical Analysis of the Integral Curves. We ignore quantities of the order O(t) in system (1.11), which corresponds formally to A = B = D = 0. Then by reducing the system to canonical form we can ascertain that it contains a singular point at $\eta = \eta_s$. The position of this point is found from the relation $\eta_s^2 = (n+1)^2 t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}} \chi(\eta_s)$. Thus, instead of (1.11), it is convenient to examine the equivalent independent system

$$\frac{dv}{d\tau} = \frac{n-1}{n+1}\eta v + \eta\frac{\partial\chi}{\partial\eta} + (2n-1)\, t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}} \chi r, \tag{2.1}$$

$$\frac{dr}{d\tau} = \frac{2n-1}{n+1} r\eta + (n-1)\, v + (n+1)\frac{\partial\chi}{\partial\eta},$$

$$\frac{d\eta}{d\tau} = \frac{1}{n+1}\eta^2 - (n+1)\, t^{\frac{2n-1}{n+1}} c^{\frac{2(2-n)}{n-1}} \chi.$$

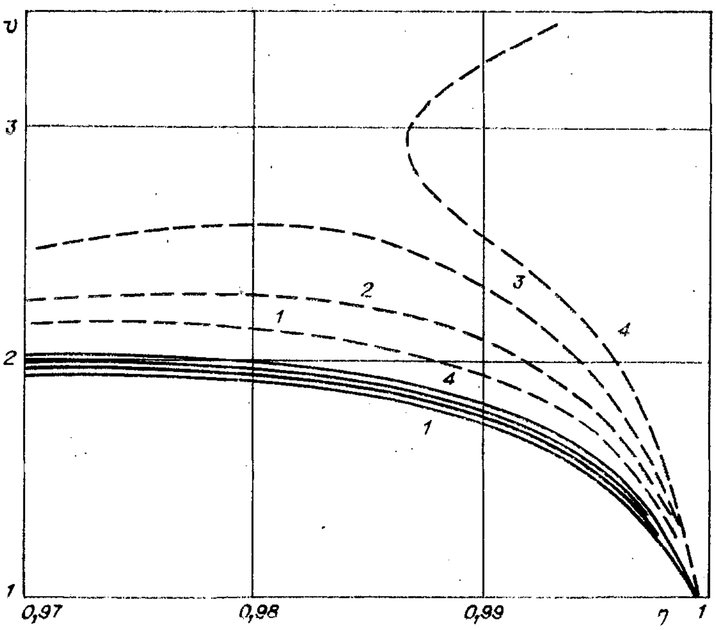

Fig. 4

Noting that the asymptote $\eta \to \eta_S$ corresponds to $\tau \to -\infty$, we divide the problem comprised of system (2.1) and conditions (1.8), (1.9) with the familiar function $\chi(\eta)$ (1.5) into two Cauchy problems in accordance with two sets of initial conditions: $v = r = 0$, $\eta = 1$ and $v = \eta = 0$, $r = r_0$. The unknown constant r_0 is determined by "joining" the solutions of both Cauchy problems at $\tau \to -\infty$.

The Cauchy problems were integrated numerically by the modified Eulerian method [9] with the step $\Delta\tau = -0.1$. Here, in the interval $0 < 1 - \eta < 10^{-4}$ the integral curve was calculated from the asymptotic representation in [10]. The calculations were repeated with a smaller step $\Delta\tau_1 = \Delta\tau/4$ to check the numerical scheme. The calculated results in Figs. 1 and 2 were used to project the integral curve on the planes $v-\eta$ and $r-\eta$ for a value of the nonlinearity index $n = 6$ (lines 1-4 correspond to $t = 0$; 0.10; 0.15; 0.20). The position of the singular point is indicated by arrows.

It can be seen that the mass of gas enveloped by the motion is gradually concentrated near the boundary of the region heated by radiation. The motion acquires the character of a shock wave [2]. Compared to [2], the present solution contains a singular point. In connection with this, we note that it is the very presence of a singular point that makes it impossible to construct continuous solutions to similarity problems (see [6, 11], for example). A similar situation arises in the present case as well.

Analysis of the integral curves obtained shows that the functions $v(\eta)$ and $r(\eta)$ undergo a removable discontinuity when crossing the singular point $\eta = \eta_S$. Beginning with the moment of time $t \approx 0.225$, a sharp point appears at $\eta = \eta_S$ in the velocity and pressure profiles (see curve 5 in Figs. 1 and 2 for $t = 0.23$), which is physically unrealistic. We should expect the occurrence of an isothermal shock wave at $t < 0.225$, while in the given approximation the solution turns out to be continuous for any moment of time t. Thus, to confirm the absence of a continuous solution, we increase the accuracy of the asymptotic representation by taking into account the initially discarded terms of the order $O(t)$. To do this, we need to calculate the differential expressions for A and B in (1.11) from the already-known first approximation as the functions $A = A_0(\tau, t)$ and $B = B_0(\tau, t)$. We then repeat the calculation of system (1.11) by the above-described method, assuming that $A = A_0(\tau, t)$ and $B = B_0(\tau, t)$.

Here, we calculated $A_0(\tau, t)$ and $B_0(\tau, t)$ by means of a two-point difference scheme. The results of the calculations are shown in Figs. 3 and 4 by dashed lines (curves 1-4 are for $t = 0.07$; 0.08; 0.09; 0.094; the solid lines correspond to the first approximation). It can be seen that the solution becomes ambiguous beginning with the moment of time $t = t^* \approx 0.092$. It follows from the physically natural requirement of nonambiguity of the integral curve that system (1.11) has no continuous solution. Evidently the solution must contain an isothermal discontinuity at $t > t^*$.

Thus, it was shown that in the asymptotic analysis of the motion of a nonlinearly heat-conducting ideal gas at $t^* \to 0$ and $n \gg 1$, occurring in a point-source thermal explosion, the appearance of a dynamic boundary layer near the boundary of the heated zone $|x| = x_f(t)$ must be taken into account even in the first approximation. In particular, this makes it impossible to construct a continuous asymptotic representation of the solution of the problem, at least when $t > t^*$. The moment of time t^* may be physically related to the moment of appearance of an isothermal shock wave.

In conclusion, we should emphasize that the above asymptotic analysis is based essentially on the physically valid assumption of the predominance of radiative heat transfer at $n \gg 1$ and $t \to 0$ [1]. The applicability of this assumption also determines the reliability of the asymptotic representations constructed.

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